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The motion of a particle in a viscous fluid under gravity, vibration and Basset's force☆

Ye.V. Visitskii *, A.G. Petrov, M.M. Shunderyuk

Moscow, Russia

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ABSTRACT

The sedimentation of a heavy solid spherical particle from a state of rest in an incompressible viscous fluid in a vessel with a vibrating bottom is investigated. Taking the Basset force into account, the problem is reduced to solving a Cauchy problem for a linear integro-differential equation. An exact solution of this problem and simple asymptotic formulae are obtained and a complete analysis of the effect of the Basset force on the oscillations and sedimentation of the particles is carried out. It is shown that a consideration of the Basset force introduces a considerable correction to the classical amplitude-frequency relation, reducing its value and also considerably slowing the arrival of the amplitude at a constant value. When there are no vibrations, it follows from the solution of the problem that there is a slow establishment of the limiting velocity (inversely proportional to the square root of the time), which differs considerably from the case of the sedimentation of particles in accordance with Stokes's law (the establishment of the limiting velocity occurs exponentially).

A force, which depends on the prehistory of the motion, acts on a solid spherical particle, moving with variable velocity in a viscous fluid. For a high viscosity fluid, it can be represented in terms of a linear integral operator, found by Basset. Basset also obtained an integro-differential equation, which the rate of sedimentation of the particle satisfies (see the references in the monograph Ref. 1).

A conversion of the integro-differential equation into an ordinary second-order differential equation was obtained in Ref. 2, describing the forced oscillations of a linear oscillator for a resistance force proportional to the velocity. The particular solution of the ordinary differential equation, which satisfies the corresponding initial conditions, is also a solution of the integro-differential equation.

In many subsequent papers (see, for example, Ref. 3) it is stated that the ordinary differential equation yields a numerical solution more simply in the sense of the amount of time and memory capacity required. However, with respect to the densities of the fluid and the particle $k = \rho/\rho_s < 4/7$ the drag, which occurs in the ordinary differential equation, changes sign and becomes a thrust, as a result of which a numerical construction of the particular solution of the ordinary differential equation becomes impossible due to the exponential increase in small perturbations. In view of this, the corresponding range of the phase densities was investigated[4] using a numerical solution of the initial integro-differential equation by the method proposed earlier in Ref. 3.

The difficulties involved in constructing a numerical solution when $k < 4/7$ have stimulated attempts to construct a particular analytical solution that can be conveniently used, corresponding to the solution of the problem of the sedimentation of the particle from a state of rest. Formal solutions, obtained by a Laplace transformation, are given in Refs 5–7. The solution, in the most compact form, is[5]

$$u(\tau) = \frac{\sqrt{k_1}}{\alpha - \beta}\left[\frac{e^{\alpha\tau}\operatorname{Erfc}\sqrt{\alpha\tau}}{\sqrt{\alpha}} - \frac{e^{\beta\tau}\operatorname{Erfc}\sqrt{\beta\tau}}{\sqrt{\beta}}\right], \quad k_1 = \frac{9\rho}{2\rho_s + \rho}, \quad u(\tau) = \frac{U_\infty - U}{U_\infty}$$

where α and β are the roots of the equation $x^2 + (2 - k_1)x + 1 = 0$, ρ is the fluid density, ρ_s is the particle density, U is the particle velocity and U_∞ is the particle steady velocity.

However, the roots α and β can be either negative or complex. It has not been stated how the real function can be separated from the formal complex solution, and hence the direct use of this solution to construct, for example, a graph of the relation $u(\tau)$ is impossible. In a

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* Corresponding author.
E-mail address: petrovipmech@gmail.com (Ye.V. Visitskii).

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commentary on the solution, presented in another form in Ref. 6, it was proposed to use the MAPLE system to separate the real branch of the solution and to construct the asymptotic form for long times, but these results were not presented.

Hence, the purpose of obtaining a solution of the problem of the sedimentation of a particle from a state of rest that is convenient for practical use, has so far not been achieved, although a qualitative result is known: the value of the particle velocity approaches its exact steady-state value at a rate that is inversely proportional to the square root of the time. Experiments[8] have shown that even for Reynolds numbers considerably exceeding unity, the root asymptotic form at the beginning of the sedimentation process holds but it then changes to an exponential law.

Below we present a simplified algorithm for reducing the integro-differential equation to an ordinary differential equation and we obtain an exact solution of the equation of motion in explicit form for all values of the governing parameters. For long times, a principal root asymptotic form is obtained with an exact coefficient as well as the following terms of the expansion, and we calculate the path which the particle must traverse in order its velocity to be established with a specified accuracy.

We separately investigate the particle motion without a vibration of the vessel and for an arbitrary frequency and amplitude of the vibration. Asymptotic solutions are obtained on the assumption that, in the course of sedimentation a large number of oscillations of the bottom occurs, the dependence of the amplitude of the particle oscillations on the ratio of the fluid and particle densities is determined, as well as on the amplitude and frequency of the vibration of the bottom. The asymptotic forms of different orders of accuracy are compared with one another and with the exact classical solution. Estimated limits, which must be imposed on the frequency and amplitude and vibration of the vessel in order for the fluid-incompressibility approximation to be applicable, are derived.

1. The equation for the sedimentation of a spherical particle

A force acts on a solid spherical particle of radius a, in an ideal incompressible fluid of density ρ at rest at infinity. The force acts from the side of the fluid and is equal to the sum of the forces related to the associated mass and the Archimedes force[1]

$$F_1 = -\frac{1}{2}\rho V\dot{U} + \rho Vg;\quad V = \frac{4}{3}\pi a^3 \tag{1.1}$$

where U is the particle velocity, g is the acceleration due to gravity and the coordinate axis is directed vertically upwards.

Additional forces act on the particle in a viscous fluid at low Reynolds number: the Stokes force $F_{\rm St}$ and the Basset force $F_{\rm B}$ (Refs 1,9,10):

$$F_{\rm St} = -6\pi\mu aU,\quad F_{\rm B} = -6a^2\pi\sqrt{\rho\mu}\hat{I}(\dot{U}) \tag{1.2}$$

where μ is the coefficient of dynamic viscosity, and $\hat{I}$ a is the integral operator

$$\hat{I}(f(t)) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{f(x)}{\sqrt{t-x}}d\tau = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} f(t-x^2)dx \tag{1.3}$$

Hence, forces F_1 (1.1), $F_{\rm St}$ and $F_{\rm B}$ (1.2) as well as the force of gravity $-\rho_s$Vg act on the particle of density ρ_s from the side of the fluid. Taking these forces into account we obtain the following Cauchy problem for calculating the sedimentation rate of a particle from the state of rest

$$(\rho + 2\rho_s)U = -2(\rho_s - \rho)g - \frac{9\mu}{a^2}U - \frac{9}{a}\sqrt{\rho\mu}\,\hat{I}(\dot{U}),\quad U(0) = 0 \tag{1.4}$$

The properties of the operator $\hat{I}$ are indicated by the equations

$$\frac{d}{dt}\hat{I}(f(t)) = \frac{f(0)}{\sqrt{\pi t}} + \hat{I}\left(\frac{df}{dt}\right),\quad \hat{I}(\hat{I}(f(t))) = \int_0^t f(x)dx \tag{1.5}$$

$$\hat{I}\left(t^{(n-2)/2}\right) = \lambda_n t^{(n-1)/2},\quad \lambda_n = \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}$$

$$\lambda_1 = \sqrt{\pi},\quad \lambda_2 = \frac{2}{\sqrt{\pi}},\quad \lambda_3 = \frac{\sqrt{\pi}}{2},\quad \lambda_4 = \frac{4}{3\sqrt{\pi}},\quad \lambda_5 = \frac{3\sqrt{\pi}}{4},\quad \lambda_6 = \frac{16}{15\sqrt{\pi}},\ldots \tag{1.6}$$

We will introduce the dimensionless variables

$$\tau = \frac{t}{t_s},\quad u = \frac{U}{U_\infty} - 1,\quad k = \frac{\rho}{\rho_s},\quad U_\infty = \frac{2(\rho-\rho_s)a^2g}{9\mu},\quad t_s = \frac{\rho_s a^2}{9\mu} \tag{1.7}$$

where t_s is the characteristic time for the particle velocity to become established, U_∞ is the steady sedimentation rate, and we will write the Cauchy problem (1.4) for the dimensionless quantity u using the two operators $\hat{A}$ and $\hat{B}$ (Refs 2–7)

$$(\hat{A} + \hat{B})u = 0,\quad u(0) = -1,\quad \hat{A} = (k+2)\frac{d}{d\tau} + 1,\quad \hat{B} = 3\sqrt{k}\,\hat{I}\frac{d}{d\tau} \tag{1.8}$$

where the initial condition corresponds to the state of rest.

The requirement that the Reynolds number should be small imposes the following limitation

$$\mathrm{Re} = \frac{U_\infty a}{\nu} \ll 1;\quad \nu = \frac{\mu}{\rho} \tag{1.9}$$

To solve problem (1.8), using property (1.6) we can construct the expansion

$$\tau \ll 1: \quad u = -1 + \frac{\tau}{k+2} - \frac{4\sqrt{k}}{\sqrt{\pi}(k+2)^2}\tau^{3/2} + \frac{4k-1}{(k+2)^3}\tau^2 + \\ + \frac{8(4-7k)\sqrt{k}}{5\sqrt{\pi}(k+2)^2}\tau^{5/2} + \frac{55k^2-50k+4}{6(k+2)^5}\tau^3 + \dots \tag{1.10}$$

Its domain of applicability is extremely small. An expansion when $\tau \gg 1$ is more interesting. This can be obtained from the exact solution of Eq. (1.8) constructed below.

Using transformations described in Appendix A, integro-differential Eq. (1.8) can be reduced to the following ordinary differential equation

$$\ddot{u} + a\dot{u} + bu = f(\tau) \\ a = \frac{4-7k}{(k+2)^2}, \quad b = \frac{1}{(k+2)^2}, \quad f(\tau) = -\frac{3}{(k+2)^2}\sqrt{\frac{k}{\pi\tau}} \tag{1.11}$$

where the dots denote derivatives with respect to the dimensionless time τ.

Corresponding to problem (1.8) we have the following initial conditions

$$u(0) = -1, \quad \dot{u}(0) = \frac{1}{k+2} \tag{1.12}$$

Ordinary differential Eq. (1.11) was apparently derived for the first time, in a somewhat different notation, by Villat[2] and was later used in Refs 3–7. It describes forced oscillations in which $a\dot{u}$ plays the role of a dissipative force. In physical problems the friction coefficient a must be positive. It is interesting that, for sufficiently light particles, when $k = \rho/\rho_S < 4/7$, the coefficient a in ordinary differential Eq. (1.11) will be negative (the force does not reduce but increases the energy). In this case ordinary differential Eq. (1.11) is specific; when $k > 4/7$ its numerical solution is impossible due to the instability of the solution for small perturbations, which leads to the need to construct a numerical solution of the corresponding integro-differential equation.[4] Below we describe a method of constructing an exact solution and its asymptotic expansion.

2. The exact solution

When $a < 0$ the particular solution of Eq. (1.11), depending on the sign of the discriminant $a^2 - 4b$, has the form

$$\lambda^2 = a^2 - 4b > 0: \\ u = -\frac{1}{\lambda}\int_\tau^\infty f(\tau')[\exp[(|a| + \lambda)(\tau - \tau')/2] - \exp[(|a| - \lambda)(\tau - \tau')/2]]d\tau' \tag{2.1}$$

$$\lambda^2 = 4b - a^2 > 0: \quad u = -\frac{2}{\lambda}\operatorname{Im}\left(\int_\tau^\infty f(\tau')\exp[(|a| + i\lambda)(\tau - \tau')/2]d\tau'\right) \tag{2.2}$$

In the handbook (Ref. 11, Section 2.36) a solution is derived only for $a > 0$; when the inequalities $a > 0$, $\lambda^2 = 4b - a^2 > 0$ are satisfied, it has the form

$$u = \frac{2}{\lambda}\operatorname{Im}\left(\int_{-\infty}^\tau f(\tau')\exp[(a + i\lambda)(\tau - \tau')/2]d\tau'\right) \tag{2.3}$$

The case when $a > 0$, $a^2 - 4b > 0$ does not arise in the problem considered.

Expressing a, b and λ in terms of k and using functions of a complex variable

$$\Phi(z) = \int_z^\infty e^{z-\zeta}\frac{d\zeta}{\sqrt{\zeta}} = 2e^z\operatorname{Erfc}(\sqrt{z}), \quad F(z) = \int_{-\infty}^z e^{\zeta - z}\frac{d\zeta}{\sqrt{\zeta}} = 2e^{-z}\operatorname{Erfi}(\sqrt{z}) - i\sqrt{\pi}e^{-z}$$

$$\operatorname{Erfc}(z) = \int_z^\infty e^{-x^2}dx, \quad \operatorname{Erfi}(z) = \int_0^z e^{x^2}dx$$

we obtain a solution in the form

$$k > \frac{8}{5}: \quad u = \frac{1}{\sqrt{\pi(5k-8)}}\left(\frac{1}{\sqrt{\zeta_+}}\Phi(\zeta_+\tau) - \frac{1}{\sqrt{\zeta_-}}\Phi(\zeta_-\tau)\right) \tag{2.4}$$

$$\frac{4}{7} < k < \frac{8}{5}: \quad u(\tau) = \frac{2}{\sqrt{\pi(8-5k)}}\operatorname{Im}\left(\frac{1}{\zeta}\Phi(\zeta^2\tau)\right)$$
$$0 < k < \frac{4}{7}: \quad u(\tau) = \frac{2}{\sqrt{\pi(8-5k)}}\operatorname{Im}\left(\frac{1}{\zeta}F(\zeta^2\tau)\right) \tag{2.5}$$

$$\zeta_\pm = \frac{1}{2(k+2)^2}\left(7k-4 \pm 3\sqrt{k(5k-8)}\right), \quad \zeta^2 = \frac{|7k-4| + i3\sqrt{k(8-5k)}}{2(k+2)^2} \tag{2.6}$$

Formulae (2.4), (2.5) and (2.6) follow from (2.1), (2.2) and (2.3) respectively.

For boundary values of the parameter k, a solution is obtained by taking the limit and has the form

$$k = \frac{4}{7}: \quad u(\tau) = \sqrt{\frac{2}{\pi}}\int_{\frac{7}{18}\tau}^{\infty}\sin\left(\frac{7}{18}\tau - \tau'\right)\frac{d\tau'}{\sqrt{\tau'}} \tag{2.7}$$

$$k = \frac{8}{5}: \quad u(\tau) = \frac{5\tau-9}{9\sqrt{\pi}}\Phi\left(\frac{5}{18}\tau\right) - \frac{1}{3}\sqrt{\frac{10}{\pi}\tau} \tag{2.8}$$

3. Asymptotic expansions

From the exact solution, using well-known expansions of the functions Erfc(z) and Erfi(z) for small and large values of the argument respectively,[12] we obtain expansions of the functions $\Phi(z)$ and $F(z)$

$$z \ll 1: \quad \Phi(z) = \sqrt{\pi}e^{z} - 2\sqrt{z}\left(1 + \frac{2z}{1\cdot 3} + \frac{(2z)^2}{1\cdot 3\cdot 5} + \ldots\right)$$
$$F(z) = -i\sqrt{\pi}e^{-z} + 2\sqrt{z}\left(1 - \frac{2z}{1\cdot 3} + \frac{(2z)^2}{3\cdot 5} + \ldots\right) \tag{3.1}$$

$$z \gg 1: \quad \Phi(z) = \frac{1}{\sqrt{z}}\left(1 - \frac{1}{2z} + \frac{1\cdot 3}{(2z)^2} - \ldots\right)$$
$$F(z) = -2i\sqrt{\pi}e^{-z} + \frac{1}{\sqrt{z}}\left(1 + \frac{1}{2z} + \frac{1\cdot 3}{(2z)^2} + \ldots\right) \tag{3.2}$$

To check the exact solution (2.4)–(2.8) we will obtain, using series (3.1), its expansion for short times and we will compare it with expansion (1.10) for the function $u(t)$. These expansions, as might have been expected, are identical.

Finally, substituting series (3.2) into the exact solution, we obtain the following asymptotic expansion

$$\tau \gg 1: \quad u(\tau) = -3\sqrt{\frac{k}{\pi\tau}}\left(1 - \frac{7k-4}{2\tau} + \frac{9}{\tau^2}(4k-1)(k-1) + \ldots\right) \tag{3.3}$$

In Fig. 1 we compare the two-term expansion (3.3) for $k = 1/4$ (the dashed curve) with the numerical calculation (the continuous curve). The dash-dot curve represents the exact solution ignoring the Basset force. On this graph the three-term expansion merges with the numerical and exact solutions. For a description of the numerical scheme see Appendix C.

4. The phase trajectory of the settling particle

The phase trajectory in the plane of the coordinate X – the velocity U, can be constructed by integrating the equation

$$dX/dt = U = U_\infty(1 + u(t/t_s))$$

The solution has the form

$$\frac{X}{a} = \frac{\mathrm{Re}}{9k}\left(\frac{t}{t_s} + \Lambda\left(\frac{t}{t_s}\right)\right), \quad \mathrm{Re} = \frac{aU_\infty}{\nu}, \quad \Lambda(\tau) = \int_0^\tau u(\tau')d\tau' \tag{4.1}$$

The asymptotic expansion of the integral $\Lambda(\tau)$ for long times can be found by integrating Eq. (1.11) in the limits $(0, \tau)$ taking conditions (1.12) into account. We obtain

$$\dot{u}(\tau) - \frac{1}{k+2} + a(u(\tau)+1) + b\Lambda(\tau) = -\frac{6}{(k+2)^2}\sqrt{\frac{k\tau}{\pi}}$$

Substituting expansion (3.3) here, we obtain

$$\tau \gg 1: \quad \Lambda(\tau) = 8k - 2 - 6\sqrt{\frac{k\tau}{\pi}}\left(1 + \frac{7k-4}{2\tau} - \frac{3(4k-1)(k-1)}{\tau^2} + \ldots\right)$$

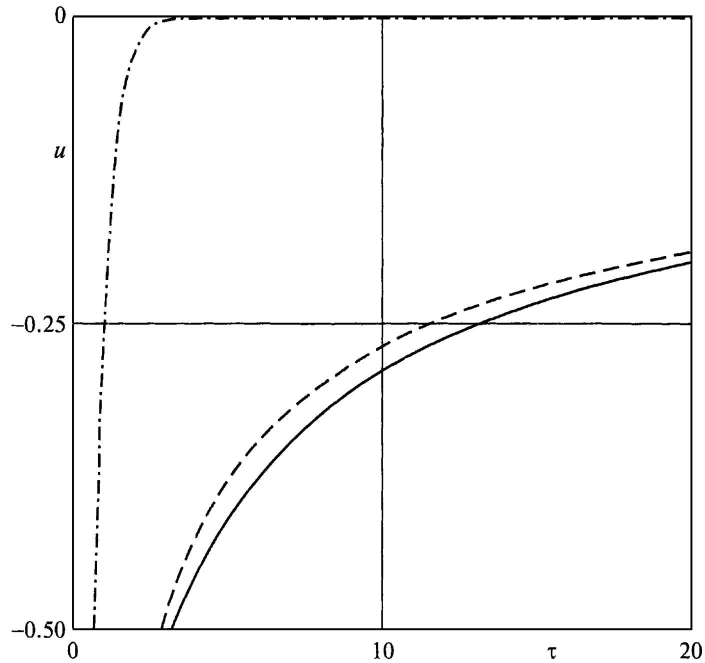

Fig. 1.

Inverting the asymptotic series u(τ) (3.3), we obtain

$$\tau(u)=\frac{9k}{\pi u^2}\left(1+\frac{\pi(-7k+4)}{9k}u^2+\frac{47k^2-64k+8}{108k^2}\pi^2u^4+\ldots\right)$$

Substituting the expansions obtained into Eq. (4.1), we have the asymptotic expansion for the phase trajectory

$$\frac{X}{a}=\mathrm{Re}\left[\frac{1}{\pi u^2}+\frac{2}{\pi u}+\frac{k+2}{9k}+\frac{47k^2-64k+8}{108k^2}\pi u^2+\ldots\right] \tag{4.2}$$

Hence, the limiting velocity U_∞ with a relative error u is established in terms of the time $t\approx a^2/(\pi u^2 \nu)$. After this time the particle traverses a distance $X\approx a\mathrm{Re}/(\pi u^2)$.

For comparison we will derive the equation of the phase trajectory ignoring the Basset force. In this case, $\hat{B}=0$ in problem (1.11), (1.12), and the solution has the form

$$u=-\exp(-\tau/(k+2))$$

We obtain for the phase trajectory

$$\frac{X}{a}=\mathrm{Re}\frac{\tau-(k+2)(1+u)}{9k}=\mathrm{Re}\frac{k+2}{9k}[-\ln(-u)-(1+u)] \tag{4.3}$$

With this assumption the velocity U_∞ is established after a time $t=t_s(k+2)\ln(-u)$ with a relative error of u. The distance traversed by the particle in this case is equal to $X\sim a\mathrm{Re}$, i.e., it is considerably less than its radius a.

5. The sedimentation of a particle in an incompressible fluid in a vessel with a vibrating bottom

Suppose the bottom of the vessel vibrates as given by the equation

$$Z_0=A_0\sin\omega t \tag{5.1}$$

The equation of motion of the particle in a system of coordinates connected with the vessel differs from Eq. (1.4) in that g is replaced by $g(t)=g+d^2Z_0/dt^2$. For $u=U/U_\infty-1$, instead of problem (1.8), we obtain

$$(\hat{A}+\hat{B})u=A\sin\omega_s\tau,\quad u(0)=-1;\quad A=A_0\omega^2/g,\quad \omega_s=\omega t_s \tag{5.2}$$

The fluid compressibility is ignored, which imposes the following limitation on the amplitude and frequency of the vibration of the bottom[10,13,14]

$$\omega^3A_0^2\ll\frac{2}{9}\frac{(k-1)(k+2)}{k^2}gc \tag{5.3}$$

where c is the velocity of sound in the fluid.

By the algorithm in Appendix A integro-differential Eq. (5.2) can be reduced to an ordinary differential equation, the right-hand side of which is then converted using the asymptotic expansion of the integral obtained in Appendix B. As a result we arrive at the following ordinary differential equation

$$\ddot{u}+\frac{4-7k}{(k+2)^2}\dot{u}+\frac{1}{(k+2)^2}u=F_1(\tau)+F_2(\tau)+F_3(\tau)$$
$$F_1(\tau)=-\frac{3}{(k+2)^2}\sqrt{\frac{k}{\pi\tau}}$$
$$F_2(\tau)=\frac{A}{(k+2)^2}\left[-3\sqrt{k\omega_s}\cos\left(\omega_s\tau-\frac{\pi}{4}\right)+\sin\omega_s\tau+(k+2)\omega_s\cos\omega_s\tau\right]$$
$$F_3(\tau)=\frac{3A\sqrt{k}}{2(k+2)^2\tau^{3/2}}+O(\tau^{-5/2}) \tag{5.4}$$

By virtue of the linearity of Eq. (5.4) its solution can be represented in the form of the sum $u=u_1(\tau)+u_2(\tau)+u_3(\tau)$, where each term u_i corresponds to the term F_i ($i=1, 2, 3$). The dimensional displacement of the particle $X=X_1+X_2+X_3$ is found from the equation

$$dX/d\tau=t_sU_\infty(1+u_1(\tau)+u_2(\tau)+u_3(\tau))$$

The functions $u_1(\tau)$ and $X_1(t/t_s)$ correspond to solutions of problem (1.11), (1.12) on the sedimentation of a particle when there are no vibrations, derived in Section 2.

The periodic solution $u_2(\tau)$ describes harmonic oscillations of the particle

$$u_2=\frac{AH(\omega_s)}{2G(\omega_s)}\cos\omega_s\tau+\frac{AJ(\omega_s)}{2G(\omega_s)}\sin\omega_s\tau=\frac{A}{2}K(\omega_s)\cos(\omega_s(\tau-\tau_0))$$
$$G(\omega_s)=(k+2)^4\omega_s^4+(k(47k-64)+8)\omega_s^2+1$$
$$H(\omega_s)=-2(k+2)^3\omega_s^3+3(k+2)^2\sqrt{2k\omega_s}\omega_s^2+(3(-7k+4)\sqrt{2k\omega_s}+4(4k+1))\omega_s-3\sqrt{2k\omega_s}$$
$$J(\omega_s)=3(k+2)^2\sqrt{2k\omega_s}\omega_s^2-4(k+2)(4k-1)\omega_s^2+3(7k-4)\sqrt{2k\omega_s}\omega_s-3\sqrt{2k\omega_s}+2$$
$$K(\omega_s)=\frac{\sqrt{H(\omega_s)^2+J(\omega_s)^2}}{G(\omega_s)},\quad \tau_0=\arcsin\frac{1}{K(\omega_s)}$$

Note that although $G(\omega_s)=0$ when $k=4/7$ and $\omega_s=7/18$, there is a finite limit of the ratios

$$H(\omega_s)/G(\omega_s)\to-1/2,\quad J(\omega_s)/G(\omega_s)\to1/2\quad\text{When}\quad k\to4/7,\quad\omega_s\to7/18$$

i.e., the amplitude remains a finite quantity.

Integrating the function $u_2(\tau)$, we obtain the displacement

$$X_2=A_B\sin\omega(t-t_0),\quad A_B=A_0(k-1)\omega_sK(\omega_s) \tag{5.5}$$

The solutions $u_3(\tau)$ and $X_3(t/t_s)$ define the corrections to the sedimentation law when there are vibrations, and they have no effect on the amplitude of the oscillations. We do not obtain them in this paper since the purpose of our research was to determine the amplitude of forced oscillations of a particle.

From expression (5.5) we obtain the amplitude coefficient (the ratio of the amplitude of the oscillation of a particle to the amplitude of the bottom vibration A_0)

$$\alpha_B=\frac{A_B}{A_0}=|k-1|\omega_sK(\omega_s)=\frac{2|k-1|}{k+2}-\frac{3\sqrt{2k}|k-1|}{(k+2)^2\sqrt{\omega_s}}+\frac{9k|k-1|}{2(k+2)^3\omega_s}+O(\omega_s^{-3/2})] \tag{5.6}$$

Ignoring the Basset force, the amplitude coefficient is obtained from the solution of Eq. (5.2) with $\hat{I}\equiv0$

$$\alpha_{St}=\frac{A_{St}}{A_0}=\frac{2|k-1|}{\sqrt{(k+2)^2+1/\omega_s^2}} \tag{5.7}$$

When there is an unlimited increase in the dimensionless frequency ω_s, both formulae (5.6) and (5.7) give a monotonic approach of the ratio of the amplitudes to a constant value $2|k-1|/(k+2)$, but, when the Basset force is taken into account, arrival at a constant value is slowed down considerably.

We chose the following values of the parameters for numerical calculations

$$\rho=1\frac{\text{г}}{\text{cm}^3},\quad\rho_s=4\frac{\text{г}}{\text{cm}^3},\quad a=20\ \mu\text{m},\quad\mu=0.01\frac{\text{г}}{\text{cms}},\quad g=980\frac{\text{cm}}{\text{s}^2}$$

Then $k=1/4$, the relation between the dimensional and dimensionless frequencies is given by the equality $\omega=5625\,\omega_s\,\text{s}^{-1}$, and Reynolds number $\text{Re}\approx0.05$, which satisfies limitation (1.9). Inequality (5.3) gives the following estimate for the amplitude of the bottom vibration:

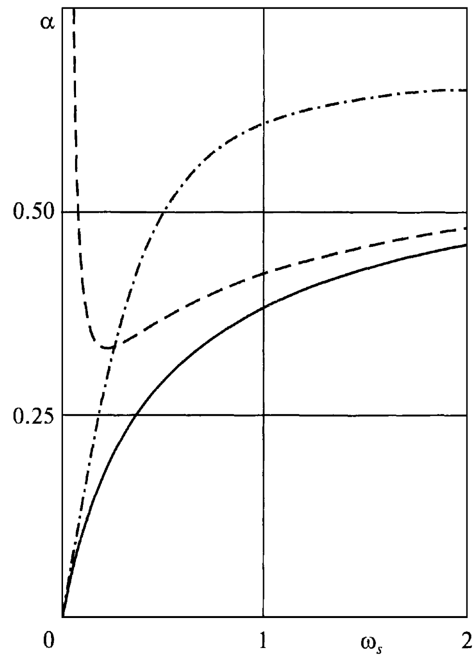

Fig. 2.

$A_0 < 0.08\ \omega_s^{-3/2}$ cm (as a result of extracting the square root we can replace the sign ≪ by the sign <). Since numerical calculations were carried out for the maximum value of the dimensionless frequency, close to 5, this estimate leads to the inequality $A_0 < 70\ \mu$m.

In Fig. 2 we show graphs of the ratio of the amplitude of the oscillations of the particle to the amplitude of the bottom vibration against the dimensionless frequency ω_s for the exact particular solution (the continuous curve), the asymptotic formula (5.6) (the dashed curve), and the solution (5.7) ignoring the Basset force (the dash-dot curve).

Hence, taking the Basset force into account we obtain a correction to the classical amplitude-frequency response, which reduces its value, and also considerably slows the arrival of the amplitude at a constant value. Graphs of $\beta = \beta(k, \omega_s) = A_{St}/A_B$ (calculated using the exact formulae) against the dimensionless frequency ω_s for different values of the parameter k are shown in Fig. 3 (sinking of a particle, $\rho < \rho_s$) and Fig. 4 (rising particle, $\rho > \rho_s$).

In Fig. 5 we show a graph of the dimensionless frequency ω_s, for which a maximum value of β is obtained for all values of ω_s, against k, and a graph of max β against k for all values of ω_s.

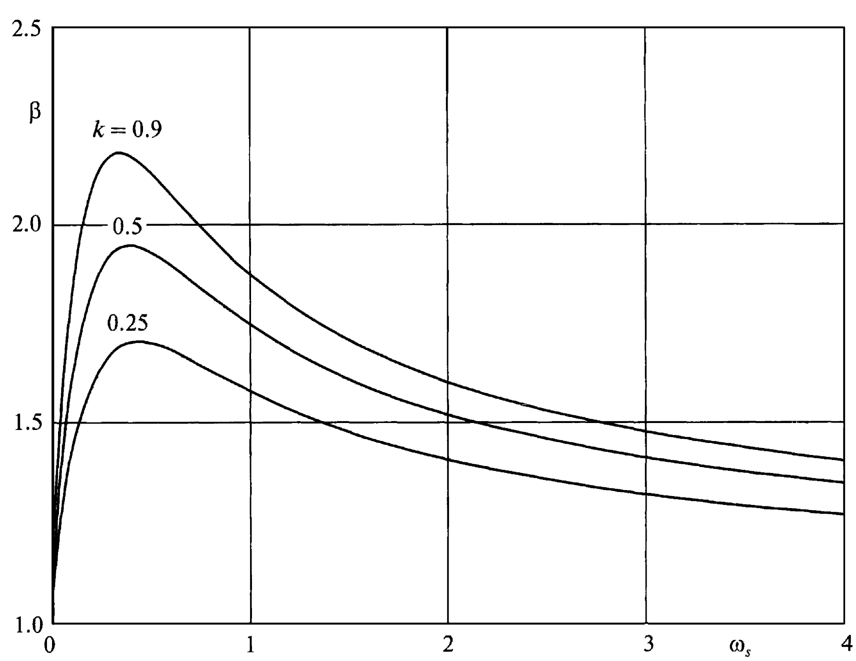

Fig. 3.

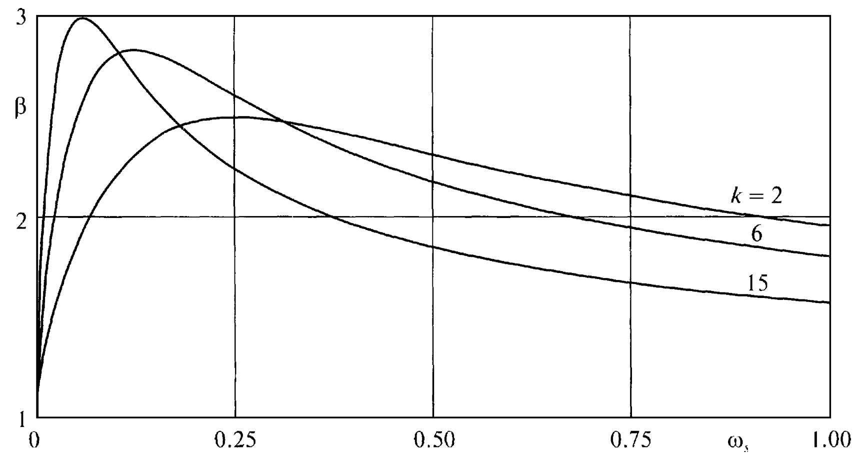

Fig. 4.

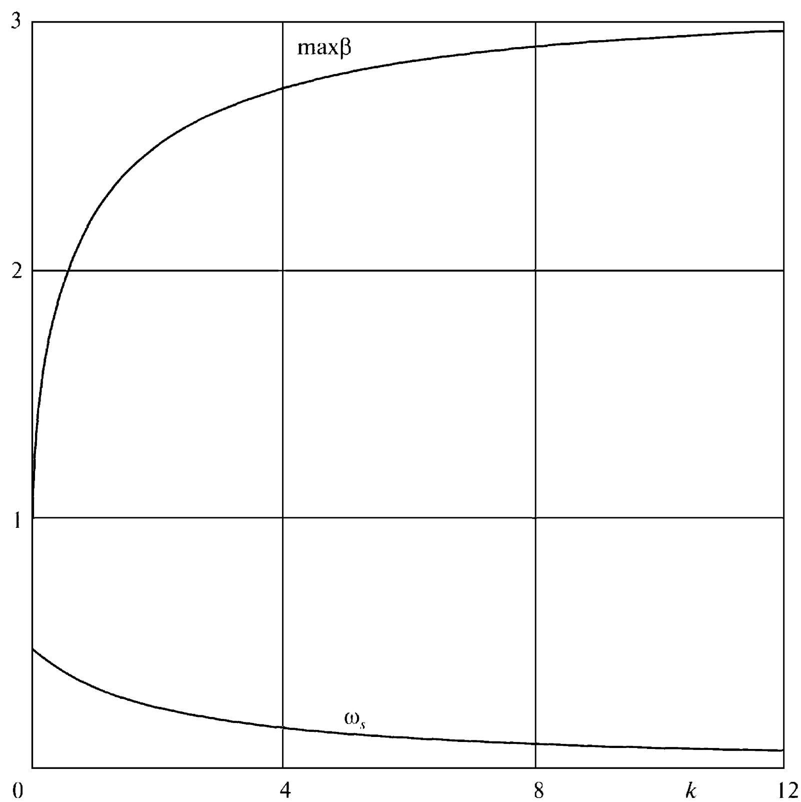

Fig. 5.

6. Conclusions

It is extremely important to take the Basset force into account when measuring the coefficient of viscosity of a fluid from the rate of sedimentation of a particle, since the limiting sedimentation rate is established after a fairly long time and the particle traverses a long path. To establish the limiting velocity of the particle with a relative accuracy $|u| = \delta U = |U - U_\infty|/U_\infty$, the particle traverses a path $X \approx a\mathrm{Re}/(\pi u^2)$ after a time X/U_∞. The path which the particle traverses to establish the limiting velocity is approximately equal to $a \times 10^4$ Re, with an accuracy to within 0.5%. For example, when measuring viscosity this path may exceed the height of the vessel in which the measurement is being carried out.

In the problem of the sedimentation of a spherical particle in a fluid with a vibrating bottom, inclusion of the Basset force introduces a correction to the classical amplitude-frequency dependence, reducing its value, and also considerably slows down the arrival of the amplitude at a constant value. For certain values of the vibration frequency and the densities of the fluid and the particle, the ratio of the amplitude of the particle to the amplitude of the bottom vibration, calculated taking the Basset force into account, when $k > 4$ is more than 2.8 times less than when the Basset force is ignored.

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Appendix A. Appendix A

Reduction of the integro-differential equation with integral operator $\hat{I}$ to an ordinary differential equation Consider the integro-differential equation of the form

$$(\hat{A}+\hat{B})u = f(\tau), \quad u(0)=u_0; \quad \hat{A}=\frac{d}{d\tau}+\gamma, \quad \hat{B}=\beta\hat{I}\left(\frac{d}{d\tau}\right) \tag{A.1}$$

Applying the operator $(\hat{A}-\hat{B})$ to both sides of the equation, we obtain the equation

$$\hat{A}^2u+(\hat{A}\hat{B}-\hat{B}\hat{A})u-\hat{B}^2u=0$$

Using properties (1.5), we arrive at the ordinary differential equation

$$\ddot{u}+(2\gamma-\beta^2)\dot{u}+\gamma^2u=-\beta\frac{\dot{u}(0)}{\sqrt{\pi\tau}}+\dot{f}(\tau)+\gamma f(\tau)-\beta\hat{I}(\dot{f}(\tau))$$

It follows from the initial equation that $\dot{u}(0)=f(0)-\gamma u_0$, and, finally, instead of system (A.1) we obtain

$$\ddot{u}+(2\gamma-\beta^2)\dot{u}+\gamma^2u=\beta\frac{\gamma u(0)-f(0)}{\sqrt{\pi\tau}}\dot{f}(\tau)+\gamma f(\tau)-\beta\hat{I}(\dot{f}(\tau))$$

$$u(0)=u_0, \quad \dot{u}(0)=f(0)-\gamma u_0$$

Appendix B. Appendix B

We will obtain the asymptotic expansion of the integral when $\tau \gg 1$

$$\int_0^\tau \frac{\cos\tau' d\tau'}{\sqrt{\tau-\tau'}}=\sqrt{\pi}\cos(\tau-\pi/4)-\frac{1}{2\tau^{3/2}}+O(\tau^{-5/2}) \tag{A.2}$$

Using replacement of the variable, the required integral is reduced to the form

$$\int_0^\tau \frac{\cos(\tau''-\tau)}{\sqrt{\tau''}}d\tau''=\chi_c(\tau)\cos\tau+\chi_s(\tau)\sin\tau; \quad \chi_c(\tau)=\int_0^\tau\frac{\cos\tau''}{\sqrt{\tau''}}d\tau'', \chi_s(\tau)=\int_0^\tau\frac{\sin\tau''}{\sqrt{\tau''}}d\tau''$$

Using integration by parts we can obtain the following expansion:

$$\chi_c(\tau)=\sqrt{\frac{\pi}{2}}+\frac{\sin\tau}{\sqrt{\tau}}-\frac{\cos\tau}{\tau^{3/2}}+O(\tau^{-5/2}); \quad \chi_s(\tau)=\sqrt{\frac{\pi}{2}}-\frac{\cos\tau}{\sqrt{\tau}}-\frac{\sin\tau}{\tau^{3/2}}+O(\tau^{-5/2})$$

Substituting these into the required integral, we obtain the required formula (A.2).

Appendix C. Appendix C

A numerical scheme for solving the equation of particle sedimentation. To solve equations of the form

$$q(t,x)\ddot{x}+p(t,x)\dot{x}=F(t,x)+s(t,x)\hat{I}(\ddot{x}-w(t,x))$$

in the WOLFRAM MATHEMATICA system we wrote a program, which uses the implicit four-point scheme of the second order of accuracy. We used the grid

$$x_i=x(t_i), \quad i=1,\ldots,n, \quad t_i=t_0+ih, \quad n=T/h$$

where T is the time for which the calculations are carried out and h is the time step.When evaluating the integral, in order to avoid singularities at the upper limit, we will use its representation in the form

$$\hat{I}(f)=\int_0^t\frac{f(t')}{\sqrt{\pi(t-t')}}dt'=-\frac{2}{\sqrt{\pi}}\int_0^t f(t')d\sqrt{t-t'}$$

The integral is replaced by the sum

$$\hat{I}(f) = -\sqrt{\frac{h}{\pi}}\sum_{k=1}^{n}\frac{f(t_k)+f(t_{k-1})}{\sqrt{n-k+1}+\sqrt{n-k}}$$

The whole scheme is written as follows (when $t_0 = 0$):

$$x_1 = x_0 + \dot{x}(0)h + \ddot{x}(0)\frac{h^2}{2}, \quad \ddot{x}(0) = -\frac{p(0,x_0)\dot{x}(0) - F(0,x_0)}{q(0,x_0)}$$

$$q(ih,x_i)\frac{x_{i+1}-2x_i+x_{i-1}}{h^2} + p(ih,x_i)\frac{x_{i+1}-x_{i-1}}{2h} = F(ih,x_i) - s(ih,x_i)\sqrt{\frac{h}{\pi}}S$$

$$S = \sum_{k=1}^{i}\frac{\ddot{x}(t_k)+\ddot{x}(t_{k-1}) - w(kh,x_k) - w((k-1)h,x_{k-1})}{\left(\sqrt{i-k+1}+\sqrt{i-k}\right)}$$

$$\ddot{x}(t_0) = \ddot{x}(0), \quad \ddot{x}(t_k) = \frac{x_{k+1}-2x_k+x_{k-1}}{h^2}$$

For each $i > 0$ the value of x_{i+1} at the next step is expressed in terms of the already-known values calculated at the previous steps. Note that the number of operations required to calculate the sum S increases in proportion to the square of the total number of points in the grid. The advantage of the scheme is its applicability to the case of both an incompressible and a compressible fluid.

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